

A generalization of κ -metrizable spaces

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December 30, 2016

Abstract

We introduce a new class of κ -metrizable spaces, namely countably κ -metrizable spaces. We show that the class of all κ -metrizable spaces is a proper subclass of countably κ -metrizable spaces. On the other hand, for pseudocompact spaces the new class coincides with κ -metrizable spaces. We prove a generalization of a Chigogidze result that the Čech-Stone compactification of a pseudocompact countably κ -metrizable space is κ -metrizable.

MSC(2010) Primary: 54B35; Secondary: 54D35,

Keywords: Čech-Stone compactification, pseudocompact, open maps, measurable cardinal, κ -metrizable spaces.

1 Introduction

All topological spaces under consideration are assumed to be at least Tychonoff.

Recall that a set $A \subseteq X$ is regular closed in a topological space X if $\text{cl int } A = A$. For a topological space X let $\text{RC}(X)$ denote the set of all regular closed sets in X and $\text{CO}(X)$ denote the set of all closed and open sets in X . The family of all complements of sets from the family $\text{RC}(X)$ forms family of regular open sets which will be denoted $\text{RO}(X)$. A topological space X is κ -metrizable if there exists function $\varrho : X \times \text{RC}(X) \rightarrow [0, \infty)$ satisfying the following axioms

(K1) $\varrho(x, C) = 0$ if and only if $x \in C$ for any $x \in X$ and $C \in \text{RC}(X)$,

(K2) If $C \subseteq D$, then $\varrho(x, C) \geq \varrho(x, D)$ for any $x \in X$ and $C, D \in \text{RC}(X)$,

(K3) $\varrho(\cdot, C)$ is a continuous function,

(K4) $\varrho(x, \text{cl}(\bigcup_{\alpha < \lambda} C_\alpha)) = \inf_{\alpha < \lambda} \varrho(x, C_\alpha)$ for any non-decreasing totally ordered sequence $\{C_\alpha : \alpha < \lambda\} \subseteq \text{RC}(X)$ and any $x \in X$.

We say that ϱ is \varkappa -metric if it satisfies conditions (K1) – (K4). The concept of a \varkappa -metrizable space was introduced by Shchepin [12]. The class of \varkappa -metrizable spaces is quite big; it contains (see e.g. [12], [13], [14])

- all metrizable spaces,
- Dugundji spaces,
- all locally compact topological group,
- the Sorgenfrey line.

Moreover

- a dense (open, regular closed) subspace of a \varkappa -metrizable space is \varkappa -metrizable,
- the product of any family of \varkappa -metrizable spaces is \varkappa -metrizable,

On the other hand, the result of Chigogidze (see [2]) implies that compactifications $\beta\omega$ and $\beta\mathbb{R}$ are not \varkappa -metrizable. Isiwata (see [7]) proved that the remainders $\beta\omega \setminus \omega$ and $\beta\mathbb{R} \setminus \mathbb{R}$ are not \varkappa -metrizable too.

If $\varrho: X \times \text{RC}(X) \rightarrow [0, \infty)$ fulfills conditions (K1) – (K3) and condition

$$(K4_\omega) \quad \varrho(x, \text{cl}(\bigcup_{n < \omega} C_n)) = \inf_{n < \omega} \varrho(x, C_n) \text{ for any chain } \{C_n : n < \omega\} \text{ and any } x \in X$$

then we say that ϱ is *countable \varkappa -metric* in X . A topological space which allows the existence of a countable \varkappa -metric we call *countably \varkappa -metrizable*.

2 Representation of countably \varkappa -metrizable spaces

Remark 2.1. Note that if X satisfies countable chain condition then every countable \varkappa -metric is \varkappa -metric.

Proof. Let $\{C_\alpha : \alpha < \lambda\} \subseteq \text{RC}(X)$ be a non-decreasing totally ordered sequence and $\lambda > \aleph_0$. By countable chain condition there exists $\alpha < \lambda$ such that $C_\beta = C_\alpha$ for all $\alpha \leq \beta$. Hence we get

$$\varrho(x, \text{cl}(\bigcup_{\alpha < \lambda} C_\alpha)) = \varrho(x, C_{\alpha+1}) = \inf_{\alpha < \lambda} \varrho(x, C_\alpha).$$

If $\lambda = \aleph_0$ then we apply condition (K4 $_\omega$). □

We give an example of countably κ -metrizable space which is not κ -metrizable space.

Let τ be an uncountable cardinal. An ultrafilter \mathcal{U} on τ is τ -complete if $\mathcal{A} \subseteq \mathcal{U}$ and $|\mathcal{A}| < \tau$, then $\bigcap \mathcal{A} \in \mathcal{U}$. An uncountable cardinal τ is *measurable* if there exists a τ -complete free ultrafilter \mathcal{U} on τ .

Let τ be an infinite cardinal and \mathcal{U} be a free ultrafilter on τ . Let $X = \tau \cup \{\mathcal{U}\}$ be space with a topology inherited from Čech-Stone compactification of τ

Lemma 2.2. *If $C \in \text{RC}(X)$, then $\mathcal{U} \in C$ if and only if $C \cap \tau \in \mathcal{U}$.*

Proof. A neighborhood of the point \mathcal{U} is of the form $D \cup \{\mathcal{U}\}$, where $D \in \mathcal{U}$. If $\mathcal{U} \in C = \text{cl int } C$ then $(\{\mathcal{U}\} \cup D) \cap \text{int } C \neq \emptyset$. Thus $D \cap C \cap \tau \neq \emptyset$ for all $D \in \mathcal{U}$, and by maximality of the filter \mathcal{U} we have $C \cap \tau \in \mathcal{U}$. If $C \cap \tau \in \mathcal{U}$ then obviously $\mathcal{U} \in \text{cl}(C \cap \tau) \subseteq C$ \square

Lemma 2.3. $\text{RC}(X) = \text{CO}(X)$

Proof. Obviously $\text{CO}(X) \subseteq \text{RC}(X)$. Let $C \in \text{RC}(X)$ and consider the following cases.

(1): $\mathcal{U} \notin C$. Then $C \subseteq \tau$ is an open subset of X .

(2): $\mathcal{U} \in C$. Then by Lemma 2.2, $C \cap \tau \in \mathcal{U}$. This finishes the proof because $C = \{\mathcal{U}\} \cup (C \cap \tau)$ is open set in X . \square

Theorem 2.4. *If τ is a measurable cardinal then the space $X = \tau \cup \{\mathcal{U}\} \subseteq \beta\tau$, where \mathcal{U} is a τ -complete free ultrafilter, is countably κ -metrizable but not κ -metrizable.*

Proof. Let $\varrho : X \times \text{RC}(X) \rightarrow \{0, 1\}$ be defined by

$$\varrho(x, C) = \begin{cases} 1 & \text{if } x \notin C, \\ 0 & \text{if } x \in C. \end{cases}$$

We claim that ϱ is countable κ -metric. Indeed, the function ϱ satisfies conditions (K1) and (K2) of definitions of κ -metric. The function $f_C = \varrho(\cdot, C)$ is continuous for any $C \in \text{RC}(X)$. Let $U \subseteq \mathbb{R}$ be an open set. If U contains 0 then $f^{-1}(U) = C$ and by Lemma 2.3 C is open set. If U contains 1 then $f^{-1}(U) = X \setminus C$ and this is open set. So, ϱ has the property (K3). It remains to verify that the function has the property (K4 $_\omega$) Let $\{C_n : n \in \omega\} \subseteq \text{RC}(X)$ be an increasing sequence and $x \in X$. If there exists $n_0 \in \omega$ such that $x \in C_{n_0}$ then we get

$$\varrho(x, \text{cl} \bigcup_{n \in \omega} C_n) = 0 = \varrho(x, C_{n_0}) = \inf\{\varrho(x, C_n) : n \in \omega\}.$$

Otherwise $x \notin C_n$ for every $n \in \omega$. If $x \notin \text{cl} \bigcup_{n \in \omega} C_n$ then obviously

$$\varrho(x, \text{cl} \bigcup_{n \in \omega} C_n) = 1 = \inf\{\varrho(x, C_n) : n \in \omega\}.$$

Let us assume, therefore, that $x \in \text{cl} \bigcup_{n \in \omega} C_n$. Then $x = \mathcal{U}$ and by Lemma 2.2, $\tau \setminus C_n \in \mathcal{U}$ for every $n \in \omega$. Since \mathcal{U} is σ -complete ultrafilter we get $D = \bigcap_{n \in \omega} (\tau \setminus C_n) \in \mathcal{U}$. Hence

$$(D \cup \{\mathcal{U}\}) \cap \bigcup_{n \in \omega} C_n = \emptyset,$$

a contradiction.

We shall prove that the space X is not κ -metrizable.

Let $C_\alpha = \alpha$ for each $\alpha < \tau$. Since $\alpha \notin \mathcal{U}$ the set C_α is clopen. Suppose that ϱ' is a κ -metric on X . Since $\mathcal{U} \in \text{cl} \bigcup \{C_\alpha : \alpha < \tau\}$ we get $\varrho'(\mathcal{U}, \text{cl} \bigcup \{C_\alpha : \alpha < \tau\}) = 0$. Therefore $\inf \{\varrho'(\mathcal{U}, C_\alpha) : \alpha < \tau\} = 0$ and there exists an increasing sequence $\{\alpha_n : n \in \omega\}$ such that $\inf \{\varrho'(\mathcal{U}, C_{\alpha_n}) : n \in \omega\} = 0$. Let $\alpha = \sup \{\alpha_n : n \in \omega\}$ or in other words $C_\alpha = \bigcup \{C_{\alpha_n} : n \in \omega\}$. Hence $0 = \inf \{\varrho'(\mathcal{U}, C_{\alpha_n}) : n \in \omega\} = \varrho'(\mathcal{U}, C_\alpha) > 0$, a contradiction. \square

A countable κ -metric $\varrho : X \times \text{RC}(X) \rightarrow \{0, 1\}$ we will call *two-valued*.

Proposition 2.5. *Assume that τ is an infinite cardinal and \mathcal{U} is a free ultrafilter on τ and $X = \tau \cup \{\mathcal{U}\} \subset \beta\tau$. If $\varrho : X \times \text{RC}(X) \rightarrow \{0, 1\}$ is a two-valued countable κ -metric then \mathcal{U} is \aleph_1 -complete on τ .*

Proof. Suppose that there exists $\{D_n : n \in \omega\} \subseteq \mathcal{U}$ and $\bigcap \{D_n : n \in \omega\} \notin \mathcal{U}$. We can assume that $D_{n+1} \subseteq D_n$ for all $n \in \omega$. Let $E_n = \text{cl} D_n$. Then $E_n \in \text{CO}(X)$ and $\mathcal{U} \in E_n$. Since $\varrho(\mathcal{U}, X \setminus E_n) = 1$ for all $n \in \omega$ and $\text{cl} \bigcup \{X \setminus E_n : n \in \omega\} = \text{cl}(X \setminus \bigcap \{E_n : n \in \omega\}) = \text{cl}(\tau \setminus \bigcap \{D_n : n \in \omega\})$ we get

$$0 = \varrho(\mathcal{U}, \text{cl} \bigcup \{X \setminus E_n : n \in \omega\}) = \inf \{\varrho(\mathcal{U}, X \setminus E_n) : n \in \omega\} = 1,$$

a contradiction. \square

Remark 2.6. Assume τ is the least cardinal that carries a two-valued countable κ -metric on $\tau \cup \{\mathcal{U}\}$. Hence τ is the least \aleph_1 -complete cardinal. By [8, Lemma 10.2] τ is measurable cardinal.

Since measurable cardinals are large cardinals whose existence cannot be proved from ZFC, it is natural to ask the question:

Question 2.7. Does there exist in ZFC a countably κ -metrizable space which is not κ -metrizable?

Assume that X is a pseudocompact space. Now we show that each κ -metrizable pseudocompact space has a special representation as an inverse limit. In order to obtain this representation we use some ideas from article [9] and monograph [6].

A continuous surjection $f : X \rightarrow Y$ is said to be *d-open* if $f[U] \subseteq \text{int cl } f[U]$ for any open set $U \subseteq X$. The notion of d-open maps was introduced by Tkachenko [15]. A function

$$f : \mathbb{R} \times \{0\} \cup \mathbb{Q} \times \{1\} \rightarrow \mathbb{R},$$

defined in the following way $f(x, 0) = x$ for any $x \in \mathbb{R}$ and $f(x, 1) = x$ for any $x \in \mathbb{Q}$ is an example of d -open but not open map. We will use the following Proposition (see [15] or [11]).

Proposition 2.8. *Let $f: X \rightarrow Y$ be a continuous function, then the following condition are equivalent:*

1. f is a d -open map,
2. there exists a base $\mathcal{B}_Y \subseteq \mathcal{T}_Y$ such that $\mathcal{P} = \{f^{-1}(V) : V \in \mathcal{B}_Y\} \subseteq \mathcal{T}_X$, i.e. for any $\mathcal{S} \subset \mathcal{P}$ and $x \notin \text{cl}_X \bigcup \mathcal{S}$, there exists $W \in \mathcal{P}$ such that $x \in W$ and $W \cap \bigcup \mathcal{S} = \emptyset$. \square

Lemma 2.9. *Let X be pseudocompact and Y be a second countable regular space and let $f: X \rightarrow Y$ be a d -open map. Then $f[\text{cl } V] = \text{cl } f[V]$ for any open subset $V \subseteq X$ and f is open map*

Proof. Let $V \subseteq X$ be an open nonempty set. It is known (see e.g. [4, Ex. 3.10.F(d)]) that $\text{cl } W$ is pseudocompact for any open nonempty set $W \subseteq X$. Since Y is separable metric space and continuous image of pseudocompact space is compact, so the image $f[\text{cl } W]$ is compact subspace for any open subset $W \subseteq X$. Therefore $\text{cl } f[V] = f[\text{cl } V]$ for any open set $V \subseteq X$. It remains to prove that f is open map. To this end, consider an open set $U \subseteq X$ and $x \in U$. There exists an open neighbourhood V of x such that $x \in V \subseteq \text{cl } V \subseteq U$. Then

$$f(x) \in f[V] \subseteq \text{int } \text{cl } f[V] = \text{int } f[\text{cl } V] \subseteq f[\text{cl } V] \subseteq f[U],$$

this completes the proof. \square

Let \mathcal{P} be a family of subsets of X . Let define an equivalent relations on X . We say that $x \sim_{\mathcal{P}} y$ if and only if

$$x \in V \leftrightarrow y \in V \text{ for every } V \in \mathcal{P}.$$

Denote by $[x]_{\mathcal{P}}$ the class of elements which is equivalent to x with respect to $\sim_{\mathcal{P}}$. By $X_{\mathcal{P}}$ we will denote a set $\{[x]_{\mathcal{P}} : x \in X\}$ and by $q: X \rightarrow X_{\mathcal{P}}$ a map $q(x) = [x]_{\mathcal{P}}$. It is clear that $q^{-1}(q(V)) = V$ for each $V \in \mathcal{P}$.

Let X be a countably κ -metrizable space with countable κ -metric ϱ . We say that $\mathcal{P} \subseteq \text{RO}(X)$ is \mathbb{Q} -admissible if it satisfies the following conditions:

- if $V \in \mathcal{P}$ then $\text{int } f_{\text{cl } V}^{-1}((-\infty, q]), \text{int } f_{\text{cl } V}^{-1}([q, \infty)) \in \mathcal{P}$ for all $q \in \mathbb{Q}$, where $f_{\text{cl } V}(\cdot) = \varrho(\cdot, \text{cl } V)$,
- $\text{int}(X \setminus V) \in \mathcal{P}$ for all $V \in \mathcal{P}$,
- $U \cap V \in \mathcal{P}$ and $\text{int } \text{cl}(U \cup V) \in \mathcal{P}$ for all $U, V \in \mathcal{P}$.

Applying inductive argument we can prove the following fact.

Lemma 2.10. For any family $\mathcal{A} \subseteq \text{RO}(X)$ there is a \mathbb{Q} -admissible family \mathcal{P} such that $\mathcal{A} \subseteq \mathcal{P}$ and $|\mathcal{P}| \leq \aleph_0 \cdot |\mathcal{A}|$. \square

Lemma 2.11. If \mathcal{P} is a \mathbb{Q} -admissible family, then for each $V \in \mathcal{P}$ there exists a sequence of increasing regular open sets $\{V_n : n \in \omega\} \subseteq \mathcal{P}$ such that

$$(*) \quad V = \bigcup_{n \in \omega} V_n \text{ and } V_n \subseteq \text{cl } V_n \subseteq V_{n+1} \subseteq V \text{ for each } n \in \omega.$$

Proof. Define $V_n = \text{int } f^{-1}([\frac{1}{n+1}, \infty])$ where $f(x) = \varrho(x, X \setminus V)$. Since $X \setminus \text{cl } V = \text{int}(X \setminus V) \in \mathcal{P}$ then $V_n \in \mathcal{P}$. It is easy to see that $V = \bigcup_{n \in \omega} V_n$ and $V_n \subseteq \text{cl } V_n \subseteq V_{n+1} \subseteq V$. \square

Lemma 2.12. If \mathcal{P} is a \mathbb{Q} -admissible family, then $[x]_{\mathcal{P}} = \bigcap \{\text{cl } V : x \in V \in \mathcal{P}\}$.

Proof. Obviously $[x]_{\mathcal{P}} \subseteq \bigcap \{\text{cl } V : x \in V \in \mathcal{P}\}$. Let $a \in \bigcap \{\text{cl } V : x \in V \in \mathcal{P}\}$. Suppose that there exists $V \in \mathcal{P}$ such that $x \in V$ and $a \notin V$. By lemma 2.11 we have $\{V_n : n \in \omega\} \subseteq \mathcal{P}$ that satisfies (*). There exists $n \in \omega$ such that $x \in V_n$. But $a \in \text{cl } V_n \subseteq V_{n+1} \subseteq V$; a contradiction. Now suppose that $a \in W$ and $x \notin W$, where $W \in \mathcal{P}$. Let $\{W_n : n \in \omega\} \subseteq \mathcal{P}$ satisfies (*) for W . There is $n \in \omega$ such that $a \in W_n$. On the other hand $x \in X \setminus \text{cl } W_n = \text{int}(X \setminus W_n) \in \mathcal{P}$, and $a \in \text{cl int}(X \setminus W_n) = X \setminus W_n$, a contradiction with $a \in W_n$. \square

Lemma 2.13. If \mathcal{P} is a countable \mathbb{Q} -admissible and $x \sim_{\mathcal{P}} y$, then $\varrho(x, \text{cl } \bigcup \mathcal{A}) = \varrho(y, \text{cl } \bigcup \mathcal{A})$ for any $\mathcal{A} \subseteq \mathcal{P}$.

Proof. Let $x, y \in X$ and $\mathcal{A} \subseteq \mathcal{P}$.

Assume first that $\mathcal{A} \subseteq \mathcal{P}$ is finite. Suppose that $\varrho(x, \text{cl } \bigcup \mathcal{A}) > \varrho(y, \text{cl } \bigcup \mathcal{A})$. There exists $q \in \mathbb{Q}$ such that $\varrho(x, \text{cl } \bigcup \mathcal{A}) \geq q > \varrho(y, \text{cl } \bigcup \mathcal{A})$. Since \mathcal{P} is \mathbb{Q} -admissible, $V = \text{int cl } \bigcup \mathcal{A} \in \mathcal{P}$ and $\text{cl } V = \text{cl } \bigcup \mathcal{A}$. For the map $f_{\text{cl } V}(\cdot) = \varrho(\cdot, \text{cl } V)$, we get $x \in \text{int } f_{\text{cl } V}^{-1}([q, \infty)) \in \mathcal{P}$ and $y \notin \text{int } f_{\text{cl } V}^{-1}([q, \infty))$, a contradiction with $x \sim_{\mathcal{P}} y$.

Assume now that \mathcal{A} is countable and infinite. We decompose \mathcal{A} into the sum of a strictly increasing sequence of families $\mathcal{A}_n \subseteq \mathcal{A}$ of strictly increasing cardinalities, $\mathcal{A} = \bigcup \{\mathcal{A}_n : n \in \omega\}$. Then

$$\varrho(x, \text{cl } \bigcup \mathcal{A}) = \inf \{\varrho(x, \text{cl } \bigcup \mathcal{A}_n) : n \in \omega\} = \inf \{\varrho(y, \text{cl } \bigcup \mathcal{A}_n) : n \in \omega\} = \varrho(y, \text{cl } \bigcup \mathcal{A})$$

by condition $(K4_{\omega})$. \square

Let X be a pseudocompact countably κ -metrizable space and \mathcal{P} be a \mathbb{Q} -admissible family. The set $X_{\mathcal{P}} = \{[x]_{\mathcal{P}} : x \in X\}$ is equipped with the topology $\mathcal{T}_{\mathcal{P}}$ generated by all images $q[V]$, $V \in \mathcal{P}$. Since \mathcal{P} is \mathbb{Q} -admissible it is closed under finite intersection and $X = \bigcup \mathcal{P}$.

Lemma 2.14 ([9, Lemma 1]). *The mapping $q : X \rightarrow X_{\mathcal{P}}$ is continuous provided \mathcal{P} is an open family X which is closed under finite intersection. Moreover, if $X = \bigcup \mathcal{P}$, then the family $\{q[V] : V \in \mathcal{P}\}$ is a base for the topology $\mathcal{T}_{\mathcal{P}}$.*

To show that if X is pseudocompact countably κ -metrizable space, then $X_{\mathcal{P}}$ is Tychonoff space, we apply the following Frink's theorem, see [5].

Theorem [O. Frink (1964)]. *A T_1 -space X is Tychonoff if and only if there exists a base \mathcal{B} satisfying:*

- (1) *If $x \in U \in \mathcal{B}$, then there exists $V \in \mathcal{B}$ such that $x \notin V$ and $U \cup V = X$;*
- (2) *If $U, V \in \mathcal{B}$ and $U \cup V = X$, then there exists disjoint sets $M, N \in \mathcal{B}$ such that $X \setminus U \subseteq M$ and $X \setminus V \subseteq N$.* \square

Theorem 2.15. *Let X be a pseudocompact countably κ -metrizable space and \mathcal{P} be countable \mathbb{Q} -admissible family. Then the $X_{\mathcal{P}}$ is compact and metrizable.*

Proof. First we shall prove that $X_{\mathcal{P}}$ is T_1 -space. Let $[x]_{\mathcal{P}} \neq [y]_{\mathcal{P}}$. Then there exists $V \in \mathcal{P}$ such that $x \in V$ and $y \notin V$. By virtue of Claim 2.11 there is a family $\{V_n : n \in \omega\} \subseteq \mathcal{P}$ which satisfies condition (*) for set V . So there is $n \in \omega$ such that $x \in V_n$. Since $y \in X \setminus \text{cl } V_n \in \mathcal{P}$ then $[y]_{\mathcal{P}} \in q[X \setminus \text{cl } V_n]$ but $[x]_{\mathcal{P}} \notin q[X \setminus \text{cl } V_n]$.

We shall prove that $X_{\mathcal{P}}$ satisfies condition (1) of Frink's theorem with a base $\mathcal{B} = \{q[V] : V \in \mathcal{P}\}$. Fix $[x]_{\mathcal{P}} \in U = q[V]$ where $V \in \mathcal{P}$. Since $V = \bigcup \{V_n : n \in \omega\}$, there exists $V_n \in \mathcal{P}$ such that $x \in V_n \subseteq \text{cl } V_n \subseteq V$. Therefore $x \notin X \setminus \text{cl } V_n \in \mathcal{P}$ and $V \cup (X \setminus \text{cl } V_n) = X$. So, $X_{\mathcal{P}} = q[V] \cup q[X \setminus V_n] = U \cup q[X \setminus V_n]$.

Let prove condition (2). Fix $U, V \in \mathcal{P}$ such that $U \cup V = X$. By Claim 2.11 there are $\{V_n : n \in \omega\} \subseteq \mathcal{P}$ and $\{U_n : n \in \omega\} \subseteq \mathcal{P}$ such that $V = \bigcup \{V_n : n \in \omega\}$ and $U = \bigcup \{U_n : n \in \omega\}$ and $V_n \subseteq \text{cl } V_{n+1}$ and $U_n \subseteq \text{cl } U_{n+1}$. Since $(X \setminus U) \cap (X \setminus V) = \emptyset$ and X is pseudocompact there exists $n \in \omega$ such that $(X \setminus \text{cl } U_n) \cap (X \setminus \text{cl } V_n) = \emptyset$ and $X \setminus U \subseteq X \setminus \text{cl } U_n \in \mathcal{P}$ and $X \setminus V \subseteq X \setminus \text{cl } V_n \in \mathcal{P}$.

By the Frink's theorem X is Tychonoff. So, X is metrizable by Urysohn Metrization theorem. Since $X_{\mathcal{P}}$ is continuous image of pseudocompact space X then X is compact. \square

Lemma 2.16. *Let X be a countably κ -metrizable pseudocompact space and \mathcal{A} be a countable family of regular open sets. There exists a d -open map $f : X \rightarrow Y$ onto a compact metrizable space with a countable base \mathcal{B} such that $\mathcal{A} \subseteq f^{-1}(\mathcal{B})$*

Proof. Let \mathcal{A} be a countable family of regular open sets and let $\mathcal{P} \subseteq \text{RO}(X)$ be a countable \mathbb{Q} -admissible family such that $\mathcal{A} \cup \{X\} \subseteq \mathcal{P}$. Let $Y = X_{\mathcal{P}}$ and $f = q : X \rightarrow X_{\mathcal{P}}$.

By Lemma 2.14 the function f is continuous and $\mathcal{A} \subseteq f^{-1}(\mathcal{B})$ for the base $\mathcal{B} = \{f[V] : V \in \mathcal{P}\}$ of Y . By Lemma 2.15 the space Y is compact and metrizable.

It remains to show that f is d -open. By Proposition 2.8 it is enough to show that for any $\mathcal{S} \subseteq \mathcal{P}$ and any $x \notin \text{cl } \bigcup \mathcal{S}$ there is $V \in \mathcal{P}$ such that $x \in V$ and $V \cap \bigcup \mathcal{S} = \emptyset$.

To do this fix arbitrary $\mathcal{S} \subseteq \mathcal{P}$ and $x \notin \text{cl} \bigcup \mathcal{S}$. We get $\varrho(x, \text{cl} \bigcup \mathcal{S}) > 0$, where ϱ is countable κ -metric on X . Hence by Lemma 2.13 $[x]_{\mathcal{P}} \cap \text{cl} \bigcup \mathcal{S} = \emptyset$. In other words, by Lemma 2.12, we have $\bigcap \{\text{cl} V : x \in V \in \mathcal{P}\} \cap \text{cl} \bigcup \mathcal{S} = \emptyset$. Since X is pseudocompact there is $V_1, \dots, V_n \in \mathcal{P}$ such that $V_1 \cap \dots \cap V_n \cap \bigcup \mathcal{S} = \emptyset$ and $x \in V_1 \cap \dots \cap V_n$. Let $V = V_1 \cap \dots \cap V_n$. The set V has required properties. \square

The notion of an almost limit was introduced by Valov [16]. We say that a space X is an *almost limit* of the inverse system $S = \{X_\sigma, \pi_\sigma^\sigma, \Sigma\}$, if there exists an embedding $q : X \rightarrow \varprojlim S$ such that $\pi_\sigma[q[X]] = X_\sigma$ for each $\sigma \in \Sigma$. We denote this by $X = a - \varprojlim S$. Obviously, if $X = a - \varprojlim S$ then X is a dense subset of $\varprojlim S$.

Theorem 2.17. *If X is pseudocompact countably κ -metrizable space, then*

$$X = a - \varprojlim \{X_\sigma, \pi_\sigma^\sigma, \Sigma\},$$

where $\{X_\sigma, \pi_\sigma^\sigma, \Sigma\}$ is a σ -complete inverse system, all spaces X_σ are compact and metrizable with countable weight, and all bonding maps π_σ^σ are open. Moreover the space $Y = \varprojlim \{X_\sigma, \pi_\sigma^\sigma, \Sigma\}$ is Čech-Stone compactification of X .

Proof. Let $\Sigma = \{\mathcal{P} \subseteq \text{RO}(X) : \mathcal{P} \text{ is a countable } \mathbb{Q}\text{-admissible family}\}$. The family Σ ordered by inclusion is directed by Lemma 2.10. Let consider $q_{\mathcal{P}} : X \rightarrow X_{\mathcal{P}}$ for each $\mathcal{P} \in \Sigma$. By Lemma 2.14 $q_{\mathcal{P}}$ is continuous map and by Lemma 2.15, each $X_{\mathcal{P}}$ is a compact metrizable space. If $\mathcal{P} \subseteq \mathcal{R}$, where $\mathcal{P}, \mathcal{R} \in \Sigma$, then we have naturally defined map $\pi_{\mathcal{P}}^{\mathcal{R}} : X_{\mathcal{R}} \rightarrow X_{\mathcal{P}}$ such that the diagram

$$\begin{array}{ccc} & X & \\ q_{\mathcal{P}} \swarrow & & \searrow q_{\mathcal{R}} \\ X_{\mathcal{P}} & \xleftarrow{\pi_{\mathcal{P}}^{\mathcal{R}}} & X_{\mathcal{R}} \end{array}$$

commutes. It is quite obvious that $\pi_{\mathcal{P}}^{\mathcal{R}}$ is open. Moreover for each increasing chain $\{\mathcal{P}_n : n \in \omega\}$ in Σ the space $X_{\mathcal{P}}$, where $\mathcal{P} = \bigcup \{\mathcal{P}_n : n \in \omega\}$, is homeomorphic to $\varprojlim \{X_{\mathcal{P}_n}, \omega\}$. So, $\{X_{\mathcal{P}}, \pi_{\mathcal{P}}^{\mathcal{R}}, \Sigma\}$ constitutes a σ -complete inverse system, where all spaces $X_{\mathcal{P}}$ are compact and metrizable and all bonding maps $\pi_{\mathcal{P}}^{\mathcal{R}}$ are open.

Now we check that the limit map $q = \varprojlim (q_{\mathcal{P}}, \Sigma) : X \rightarrow \varprojlim \{X_{\mathcal{P}}, \pi_{\mathcal{P}}^{\mathcal{R}}, \Sigma\}$ is an embedding. If $x, y \in X$ and $x \neq y$ then we can find a disjoint regular open sets U, V such that $x \in U$ and $y \in V$. By Lemma 2.10 there is $\mathcal{P} \in \Sigma$ such that $U, V \in \mathcal{P}$. So, $q(x) \neq q(y)$ because $[x]_{\mathcal{P}} \neq [y]_{\mathcal{P}}$. It remains to prove that $q[V]$ is open in $q[X]$ whenever $V \subseteq X$ is open. We can assume that $V \in \text{RO}(X)$. Let $\mathcal{P} \in \Sigma$ be such a family that $V \in \mathcal{P}$. Note that $q[V] = q[X] \cap \pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[V])$, where $\pi_{\mathcal{P}}$ denotes projection from the inverse limit to $X_{\mathcal{P}}$. Let $a \in q[X] \cap \pi_{\mathcal{P}}^{-1}[q_{\mathcal{P}}[V]]$. Then $a = q(x)$ for some $x \in X$ and hence

$$q_{\mathcal{P}}(x) = q(x)_{\mathcal{P}} = \pi_{\mathcal{P}}(q(x)) = \pi_{\mathcal{P}}(a) \in q_{\mathcal{P}}[V].$$

It means that $[x]_{\mathcal{P}} = [y]_{\mathcal{P}}$ for some $y \in V$. Therefore $x \in V$ and $a = q(x) \in q[V]$. An inclusion $q[V] \subseteq q[X] \cap \pi_{\mathcal{P}}^{-1}(q_{\mathcal{P}}[V])$ is obvious.

Claim 2.18. *For any continuous map $h : X \rightarrow [0, 1]$ there exists $\mathcal{P} \in \Sigma$ and a continuous map $g : X_{\mathcal{P}} \rightarrow [0, 1]$ such that $h = g \circ \pi_{\mathcal{P}} \upharpoonright X$.*

Let $h : X \rightarrow [0, 1]$ and $a \in [0, 1]$. There exists a sequence of open subsets $\{U_n^a : n \in \omega\}$ such that $U_{n+1}^a \subseteq \text{cl } U_{n+1}^a \subseteq U_n^a \subseteq [0, 1]$ and $\{a\} = \bigcap_{n \in \omega} U_n^a$. Since $[0, 1]$ has a countable base \mathcal{B} we may assume that $\{U_n^a : n \in \omega\} \subseteq \mathcal{B}$. Therefore we get

$$h^{-1}(U_{n+1}^a) \subseteq \text{int cl } h^{-1}(U_{n+1}^a) \subseteq \text{cl } h^{-1}(U_{n+1}^a) \subseteq h^{-1}(\text{cl } U_{n+1}^a) \subseteq h^{-1}(U_n^a)$$

and

$$h^{-1}(\{a\}) = \bigcap_{n \in \omega} \text{int cl } h^{-1}(U_n^a).$$

There exists $\mathcal{P} \in \Sigma$ such that $\{\text{int cl } h^{-1}(U_n^a) : n \in \omega, a \in [0, 1]\} \subseteq \mathcal{P}$. Now we shall prove that $h(x_1) = h(x_2)$, whenever $x_1 \sim_{\mathcal{P}} x_2$. Let $y = h(x_1)$. Then $x_1 \in \bigcap_{n \in \omega} \text{int cl } h^{-1}(U_n^y) = h^{-1}(y)$ and $\{\text{int cl } h^{-1}(U_n^y) : n \in \omega\} \subseteq \mathcal{P}$. Since $x_1 \sim_{\mathcal{P}} x_2$, we get $x_2 \in \text{int cl } h^{-1}(U_n^y)$ for every $n \in \omega$, what implies $x_2 \in h^{-1}(y)$. Define a map g by the formula $g([x]_{\mathcal{P}}) = h(x)$ for any $x \in X$. In order to show that g is continuous we will prove that $g^{-1}(U) = \pi_{\mathcal{P}}(h^{-1}(U))$ for any open subset $U \subseteq [0, 1]$. We have the following equivalence

$$[x]_{\mathcal{P}} \in g^{-1}(U) \Leftrightarrow g([x]_{\mathcal{P}}) \in U \Leftrightarrow h(x) \in U \Leftrightarrow x \in h^{-1}(U).$$

Since $h(x_1) = h(x_2)$ whenever $x_1 \sim_{\mathcal{P}} x_2$, we get $x \in h^{-1}(U) \Leftrightarrow [x]_{\mathcal{P}} \in \pi_{\mathcal{P}}(h^{-1}(U))$. This completes the proof of the claim.

Given a continuous map $h : X \rightarrow [0, 1]$ by the Claim there exists $\mathcal{P} \in \Sigma$ and a continuous map $g : X_{\mathcal{P}} \rightarrow [0, 1]$ such that $h = g \circ \pi_{\mathcal{P}} \upharpoonright X$. The map $g \circ \pi_{\mathcal{P}} : Y \rightarrow [0, 1]$ is required extension of h , hence $Y = \beta X$. \square

By the result of Kucharski [10] (see also [1]) and Theorem 2.17 we get the following Corollary, which generalizes Chigogidze's result [2, Corollary 2] about pseudocompact κ -metrizable space.

Corollary 2.19. *Any pseudocompact countably κ -metrizable space is ccc.*

For example ω_1 with the order topology is not countably κ -metrizable since it is pseudocompact and is not ccc.

We have proved that for ccc spaces or pseudocompact spaces, countably κ -metrizable spaces coincides with κ -metrizable spaces.

Question 2.20. For which class of spaces does countably κ -metrizability coincide with κ -metrizability?

3 Čech-Stone compactification of κ -metrizable space

A. Chigogidze announced in [2] without a proof that Čech-Stone compactification of κ -metrizable space is κ -metrizable. Next G. Dimov gave in [3] sufficient and necessary conditions for compact Hausdorff extension of κ -metrizable space to be κ -metrizable. In this section we will give a simple proof that Čech-Stone compactification of pseudocompact countably κ -metrizable space is κ -metrizable.

Firstly, note the following simple observation.

Lemma 3.1. *Let X be completely regular space. If $F \in \text{RC}(\beta X)$ then $F \cap X \in \text{RC}(X)$. \square*

All topological spaces X considered below are assumed to be pseudocompact and countably κ -metrizable, with countable κ -metric ϱ . So, each continuous function $\varrho(\cdot, F \cap X): X \rightarrow [0, \infty)$ is bounded say by $b_F \in [0, \infty)$ for each regular closed subset $F \subseteq \beta X$. Hence we can extend each function $\varrho(\cdot, F \cap X)$ to continuous function $\bar{\varrho}(\cdot, F \cap X): \beta X \rightarrow [0, b_F]$. Now we shall prove that a function $\psi: \beta X \times \text{RC}(\beta X) \rightarrow \mathbb{R}$ defined by the formula

$$\psi(p, F) = \bar{\varrho}(p, F \cap X)$$

is countable κ -metric or satisfies condition (K1), (K2) and (K4 $_\omega$) (the condition (K3) is obviously fulfilled).

Claim 3.2 (K1). *Let $F' \in \text{RC}(\beta X)$ and $F = F' \cap X$.*

1. *If $p \in F'$ then $\bar{\varrho}(p, F) = 0$.*

2. *if $\bar{\varrho}(p, F) = 0$, then $p \in F'$.*

Proof. (1) Suppose that $\bar{\varrho}(p, F) > 0$. Then there is an open neighbourhood $V \subseteq \beta X$ of p and $b > 0$ such that $\bar{\varrho}(q, F) > b$ for each $q \in V$. Since $V \cap \text{int}_{\beta X} F' \neq \emptyset$ there is $y \in X \cap V \cap \text{int}_{\beta X} F'$. Therefore $b < \bar{\varrho}(y, F) = \varrho(y, F) = 0$, a contradiction.

(2) Suppose that $p \notin F'$. There are open neighborhoods $V_n \subseteq \beta X$ of p such that $\bar{\varrho}(y, F) < \frac{1}{n}$ for $y \in V_n$, $V_n \cap F' = \emptyset$ and $\text{cl}_{\beta X} V_{n+1} \subseteq V_n$. Since X is pseudocompact space $\bigcap_{n \in \omega} \text{cl}_{\beta X} V_n \cap X \neq \emptyset$. Let $y_0 \in \bigcap_{n \in \omega} \text{cl}_{\beta X} V_n \cap X$. Therefore $\frac{1}{n} \geq \bar{\varrho}(y_0, F) = \varrho(y_0, F) > 0$ for each $n \in \omega$, a contradiction. \square

Claim 3.3 (K2). *Let $F', G' \in \text{RC}(\beta X)$ and $F = F' \cap X$ and $G = G' \cap X$. If $F' \subseteq G'$, then $\bar{\varrho}(p, F) \geq \bar{\varrho}(p, G)$ for every $p \in \beta X$.*

Proof. Suppose that there exists $p \in \beta X$ such that $\bar{\varrho}(p, F) < \bar{\varrho}(p, G)$. There is a neighborhood V of p such that $\bar{\varrho}(q, F) < \bar{\varrho}(q, G)$ for every $q \in V$. Let $a \in V_x \cap X$, then $\varrho(a, F) = \bar{\varrho}(a, F) < \bar{\varrho}(a, G) = \varrho(a, G)$, a contradiction. \square

Claim 3.4 ($K4_\omega$). Let $\{F'_n : n \in \omega\} \subseteq \text{RC}(\beta X)$ be a such family that $F'_n \subseteq F'_{n+1}$ and let $F_n = F'_n \cap X$. Then $\bar{\varrho}(p, \text{cl}_{\beta X}(\bigcup_{n \in \omega} F_n) \cap X) = \inf\{\bar{\varrho}(p, F_n) : n \in \omega\}$ for every $p \in \beta X$.

Proof. Let $F = \text{cl}_{\beta X}(\bigcup_{n \in \omega} F_n) \cap X$. By Claim 3.3 we get an inequality " \leq ". Suppose that $\bar{\varrho}(x, F) < b < \inf\{\bar{\varrho}(x, F_n) : n \in \omega\} = a$. There exists neighbourhood V of x such that $\bar{\varrho}(y, F) < b$ for every $y \in V$. There is a sequence $\{V_n : n \in \omega\}$ of open sets of βX such that $\text{cl}_{\beta X} V_{n+1} \subseteq V_n \subseteq \text{cl}_{\beta X} V_n \subseteq V$ and $\bar{\varrho}(y, F_n) \in (a - \frac{1}{n}, a + \frac{1}{n})$ for every $y \in V_n$. Since X is pseudocompact there is $y \in \bigcap_{n \in \omega} \text{cl}_{\beta X} V_n \cap X$. Therefore

$$b < a = \inf\{\bar{\varrho}(y, F_n) : n \in \omega\} = \inf\{\varrho(y, F_n) : n \in \omega\} = \varrho(y, F) = \bar{\varrho}(y, F) < b,$$

a contradiction. \square

Theorem 3.5. If X is pseudocompact countably κ -metrizable space then βX is κ -metrizable.

Proof. We use Remark 2.1, Corollary 2.19 and previous Claims. \square

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